4.2 The definite integral

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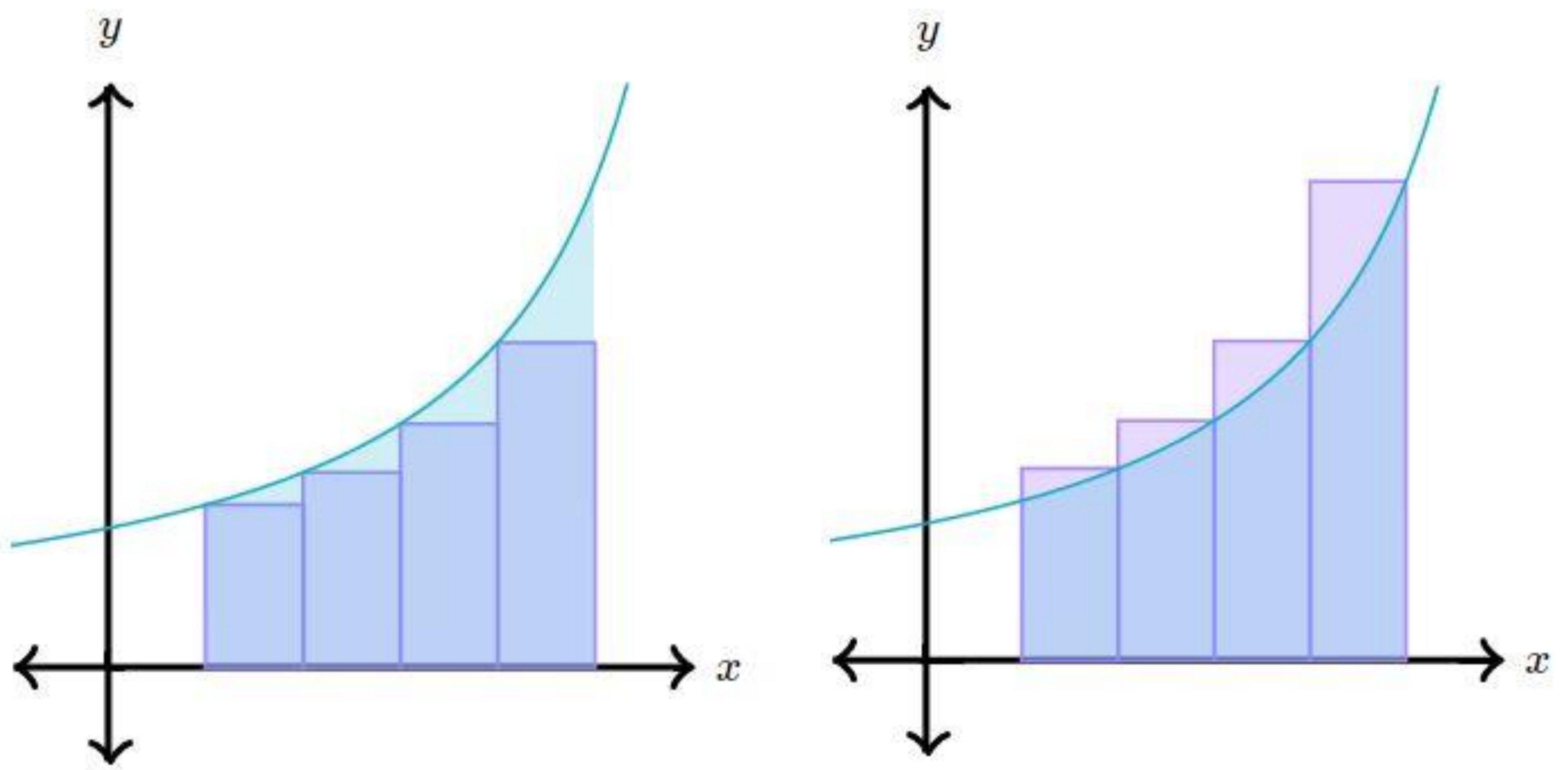
Reference: Varberg, Purcell, Rigdon (ninth edition) . *Calculus*.

Riemann sum

$$\mathrm{R}_P = \sum_{i=1}^{n} f(\bar{x}_i).\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i).\,\Delta x$$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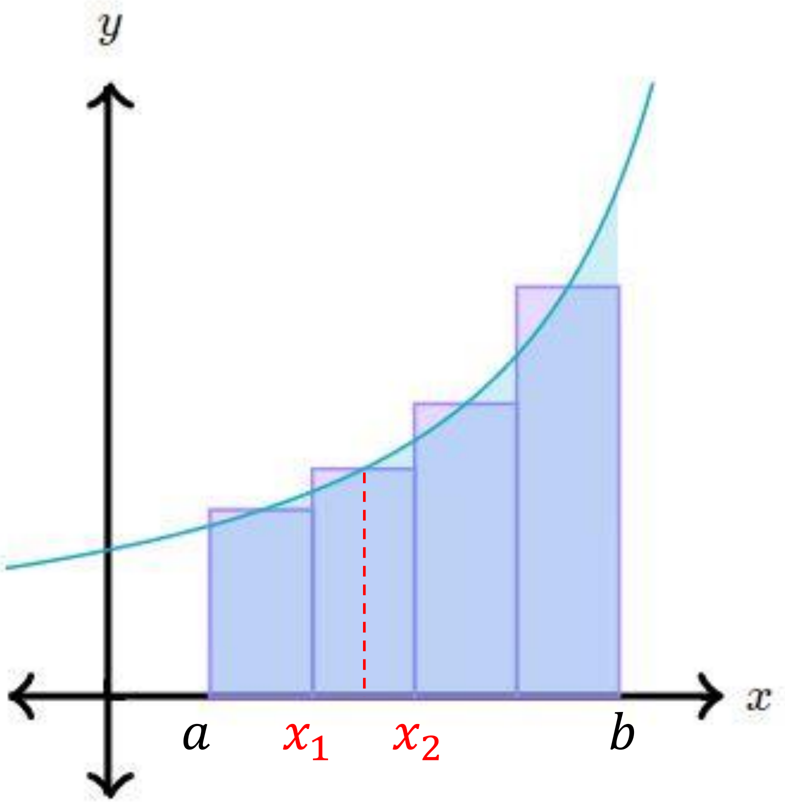


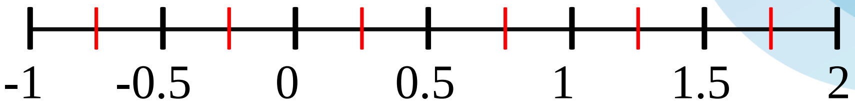
$$[a, b]$$

$$interval\ length: \quad b - a$$

$$midpoint: \quad \bar{x}_i = \frac{x_1 + x_2}{2}$$

$$\Delta x_i = \frac{b - a}{n}$$

Example 1: Evaluate the Riemann sum for $f(x) = x^2 + 1$ on the interval [-1,2] using the equally spaced partition points $-1 < -0.5 < 0 < 0.5 < 1 < 1.5 < 2$, with the sample point $\bar{x}_i$ being the midpoint of the ith subinterval.



$$\bar{x}_1 = \frac{-1 + (-0.5)}{2} = -0.75 \qquad \bar{x}_2 = \frac{-0.5 + 0}{2} = -0.25$$

$$\bar{x}_3 = \frac{0 + 0.5}{2} = 0.25 \qquad \bar{x}_4 = \frac{0.5 + 1}{2} = 0.75$$

$$\bar{x}_5 = \frac{1 + 1.5}{2} = 1.25 \qquad \bar{x}_6 = \frac{1.5 + 2}{2} = 1.75$$

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$$\Delta x_i = \frac{2-(-1)}{6} = 0.5$$

$$\mathrm{R}_P = \sum_{i=1}^{6} f(\overline{x_i}).\Delta x_i$$

$$= [f(-0.75)(0.5) + f(-0.25)(0.5) + f(0.25)(0.5) + f(0.75)(0.5)$$

$$+ f(1.25)(0.5) + f(1.75)(0.5)]$$

$$= (0.5)[f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)]$$

$$= (0.5)[1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625]$$

$$= (0.5)[11.875] = 5.9375$$

Example 2: Evaluate the Riemann sum R_P for

$$f(x) = (x+1)(x-2)(x-4) = x^3 - 5x^2 + 2x + 8$$

on the interval $[0,5]$ using the partition $P \ldots$

$$0 < 1.1 < 2 < 3.2 < 4 < 5$$

$$\bar{x}_1 = 0.5 \qquad \bar{x}_2 = 1.5 \qquad \bar{x}_3 = 2.5 \qquad \bar{x}_4 = 3.6 \qquad \bar{x}_5 = 5$$

$$\mathrm{R}_P = \sum_{i=1}^{5} f(\overline{x_i}).\Delta x_i$$

$$= [f(0.5)(1.1) + f(1.5)(0.9) + f(2.5)(1.2) + f(3.6)(0.8) + f(5)(1)]$$

$$= 23.9698$$

The indefinite integral

$$\int f(x)\,dx = F(x) + c$$

The definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} R_P = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\bar{x}_i).\Delta x_i$$

norm $||P||$: the length of the longest of the subintervals of the partition P

$$\lim_{||P||\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Riemann integral steps:

1- Partition the interval into n equal subintervals.

2- Take any point and substitute in the function.

3- Find Riemann sum

4- Take the limit as as $n \to \infty$

Example 3: Evaluate $\int_{-2}^{3}(x+3)dx$.

$$\Delta x_i = \frac{3-(-2)}{n} = \frac{5}{n} \qquad [-2,3]$$

$$x_0 = -2 + (0)\Delta x \qquad x_1 = -2 + (1)\Delta x \qquad x_2 = -2 + (2)\Delta x$$

$$x_i = -2 + (i)\Delta x = -2 + \frac{5i}{n}$$

$$f(x_i) = x_i + 3 = -2 + \frac{5i}{n} + 3 = 1 + \frac{5i}{n}$$

$$R_P = \sum_{i=1}^{n} f(x_i)\ \Delta x = \sum_{i=1}^{n} \left[1 + \frac{5i}{n}\right]\left(\frac{5}{n}\right)$$

$$R_P = \sum_{i=1}^{n} \left[1 + \frac{5i}{n}\right]\left(\frac{5}{n}\right) = \sum_{i=1}^{n} \frac{5}{n} + \sum_{i=1}^{n} \frac{25i}{n^2}$$

$$= \frac{5}{n}n + \frac{25}{n^2}\sum_{i=1}^{n} i = 5 + \frac{25}{n^2}\left(\frac{n(n+1)}{2}\right) = 5 + \frac{25}{2}\left(\frac{n^2+n}{n^2}\right)$$

$$= 5 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = 5 + \frac{25}{2} + \frac{25}{2\mathrm{n}}$$

$$\int_{-2}^{3} (x+1)\,dx = \lim_{n\to\infty}\left(5 + \frac{25}{2} + \frac{25}{2\mathrm{n}}\right) = 5 + \frac{25}{2} = \frac{35}{2}$$

Example 4: Evaluate $\int_{-1}^{3}(2x^2-8)dx$.

$$\Delta x_i = \frac{3-(-1)}{n} = \frac{4}{n} \qquad [-1,3]$$

$$x_0 = -1+(0)\Delta x \qquad x_1 = -1+(1)\Delta x \qquad x_2 = -1+(2)\Delta x$$

$$x_i = -1+(i)\Delta x = -1+\frac{4i}{n}$$

$$f(x_i) = 2x_i^2 - 8 = 2\left(-1+\frac{4i}{n}\right)^2 - 8 = 2\left(1-\frac{8i}{n}+\frac{16i^2}{n^2}\right) - 8 = -6-\frac{16i}{n}+\frac{32i^2}{n^2}$$

$$R_P = \sum_{i=1}^{n} f(x_i)\ \Delta x = \sum_{i=1}^{n}\left[\frac{32i^2}{n^2}-\frac{16i}{n}-6\right]\left(\frac{4}{n}\right)$$

$$R_P=\sum_{i=1}^{n}\left[\frac{32i^2}{n^2}-\frac{16i}{n}-6\right]\left(\frac{4}{n}\right)=\frac{128}{\mathrm{n}^3}\sum_{i=1}^{n}i^2-\frac{64}{n^2}\sum_{i=1}^{n}i-\sum_{i=1}^{n}\frac{24}{n}$$

$$=\frac{128}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)-\frac{64}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{24n}{n}$$

$$=\frac{128}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right)-\frac{64}{2}\left(\frac{n^2+n}{n^2}\right)-24=\frac{64}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)-32\left(1+\frac{1}{n}\right)-24$$

$$=\frac{128}{3}+\frac{64}{n}+\frac{64}{3n^2}-32+\frac{32}{\mathrm{n}}-24=\frac{64}{3\mathrm{n}^2}+\frac{64}{\mathrm{n}}-\frac{40}{3}$$

$$\int_{-1}^{3}(2x^2-8)\,dx=\lim_{n\to\infty}\left(\frac{64}{3\mathrm{n}^2}+\frac{64}{\mathrm{n}}-\frac{40}{3}\right)=-\frac{40}{3}$$

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